Logic in Dependent Type Theory

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Special Session in memoriam of Peter Aczel

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Lucky coincidences

- Peter spent 1997/98 on sabbatical, 1st semester at the University of Nijmegen and 2nd semester at the University of Padua.
- Manchester and Padua had an Erasmus agreement, which allowed me to spend 1998/99 at Manchester.
- PhD at Manchester (January 2000 - December 2002).

Some of Peter Aczel's main contributions

- PhD thesis, University of Oxford (1966)
- Theory of inductive definitions (1977)
- Type-theoretic interpretation of CST (1977, 1982, 1986)
- Frege structures (1980)
- Non-well-founded sets (1988)
- Final coalgebra theorem (1989)
- Pointfree topology in CZF (1998 onwards)

This talk: 1997-2013

Peter's work during this period played an important role in 'liberating' type theory from the propositions-as-types representation of logic, thus opening the way for HoTT.

I will try to reconstruct this fragment of history.

- **Part I.** Relating sets and types
- **Part II.** Logic-enriched type theories
- **Part III.** Towards HoTT

Part I:
Relating types and sets

Proof Theory on the eve of the year 2000

> *"Hilbert's metamathematics provides an alternative approach to the philosophical one to comparing belief systems for mathematics: capture the ideas of each belief system in a formal system and study and compare the formal systems and in particular their consistency sentences either using finitist methods or else perhaps using stronger methods that one believes in.*
>
> *Proof Theory should provide the tools to do this!"*

P. Aczel, email to Sol Feferman, October 1999.

Types and sets

We will consider

- extensions of CZF, e.g. IZF.
- extensions of Martin-Löf type theory:

$$0\,, \quad 1\,, \quad \mathrm{Bool}\,, \quad \mathrm{N}\,, \quad \mathrm{Id}_A(a,b)$$

$$A \times B\,, \quad A + B\,, \quad A \to B\,,$$

$$(\Sigma x\!:\!A)B(x)\,, \quad (\Pi x\!:\!A)B(x)\,, \quad (Wx : A)B(x)\,,$$

$$\mathrm{U}_n$$

E.g. $\mathrm{MLWU}_{<\omega}$ has U_n for all n, reflecting all types.

The original problem

Talk in Castiglioncello 1998.

Working definitions: a theory T is

- *generalised predicative* if

$$T \leq \mathrm{MLWU}_{<\omega}$$

- *fully impredicative* if

$$\mathrm{PA}^2 \leq T$$

Questions:

1. Can we understand fully impredicative theories constructively?
2. What is the strength of the Calculus of Inductive Constructions?

The strength of Martin-Löf type theory

Theorem (Aczel 1999)**.** There is a proof-theoretic equivalence:

$$\mathsf{MLWU}_{\leq\omega} \equiv \mathsf{CZF}^{+}\mathsf{u}_{\leq\omega}$$

Idea:

- types-as-sets interpretation

$$\mathsf{MLWU}_{n} \leq \mathsf{CZF}^{+}\mathsf{u}_{n}$$

- sets-as-trees interpretation

$$\mathsf{CZF}^{+}\mathsf{u}_{n} \leq \mathsf{MLWU}_{n+1}$$

- the interpretations catch up with each other at ω.

Problem. Extend this to fully impredicative type theories.

Full impredicativity in type theory (I)

A version of Russell's reducibility axiom:

$$\mathsf{P} : \mathsf{U} \qquad\qquad \frac{A : \mathsf{P}}{\mathsf{T}(A) : \mathsf{U}}$$

$$\frac{A : \mathsf{type}}{\langle A \rangle : \mathsf{P}} \qquad\qquad \frac{A : \mathsf{type}}{\mathsf{r}_A : \mathsf{T}(A) \leftrightarrow \langle A \rangle}$$

These imply validity of Power Set and Full Separation (Aczel 1986).

Note. Cf. *generic proof* (Menni 2003), used to characterise categories whose exact completions are toposes.

Full impredicativity in type theory (II)

A version of the Calculus of Constructions (Coquand & Huet)

$$\mathrm{Prop} : \mathrm{type} \qquad \frac{A : \mathrm{Prop}}{\mathrm{T}(A) : \mathrm{type}}$$

$$\frac{A : \mathrm{type} \qquad x : A \vdash B(x) : \mathrm{Prop}}{(\forall x : A) B(x) : \mathrm{Prop}} \qquad \frac{A : \mathrm{type} \qquad x : A \vdash B(x) : \mathrm{Prop}}{\pi_{A,B} : \mathrm{T}((\forall x : A) B(x)) \leftrightarrow (\Pi x : A) \mathrm{T}(B(x))}$$

$$\bot : \mathrm{Prop} \qquad \tau : \mathrm{T}(\bot) \leftrightarrow 0$$

Note. Let $A = \mathrm{Prop}$, to get

$$\frac{x : \mathrm{Prop} \vdash B(x) : \mathrm{Prop}}{(\forall x : \mathrm{Prop}) B(x) : \mathrm{Prop}}$$

Cf. Russell's vicious circle paradox.

Two treatments of logic

(1) **Curry-Howard:** propositions-as-types

(2) **Russell-Prawitz:** propositions-as-types-in-Prop, i.e. for $A, B : \mathrm{Prop}$

$$\begin{aligned} \bot &= (\forall x : \mathrm{Prop})x \\ A \wedge B &= (\forall x : \mathrm{Prop})\big(A \to (B \to x)\big) \\ A \vee B &= (\forall x : \mathrm{Prop})\big((A \to x) \to \big((B \to x) \to x\big)\big) \\ \neg A &= (\forall x : \mathrm{Prop})(A \to x) \end{aligned}$$

and for $A : \mathrm{type}$ and $x : A \vdash B(x) : \mathrm{Prop}$,

$$(\exists x : A)B(x) = (\forall u : \mathrm{Prop})\big((\forall x : A)(B(x) \to u) \to u\big)$$

Cf. Girard's System F.

The Russell-Prawitz modality

For A : type, define $J(A)$: Prop by

$$J(A) = (\forall x : \mathrm{Prop})\big((A \to x) \to x\big)$$

Proposition (Aczel 1999)**.**

$$\begin{aligned} \bot &\leftrightarrow J(0) \\ A \wedge B &\leftrightarrow J(A \times B) \\ A \vee B &\leftrightarrow J(A + B) \\ \neg A &\leftrightarrow J(\neg A) \\ (\exists x : A)B &\leftrightarrow J((\Sigma x : A)B(x)) \end{aligned}$$

So the Russell-Prawitz treatment of logic arises from the Curry-Howard treatment via J.

Question. What is J?

$\mathbb{P}$ is a type operator T such that

For each type A

$TA : Prop$

$A \to TA$

$(A \to B) \to (TA \to B)$ for $B : Prop$

$T\mathbb{N}_0 \to \mathbb{N}_0$

or equivalently

① T is a <u>lax type operator</u>

i.e. $A \to TA$

$TTA \to TA$

$(A \to A') \to (TA \to TA')$

for all types A, A'

② T is <u>strict</u> i.e. $T\mathbb{N}_0 \to \mathbb{N}_0$

③ $Prop$ is a <u>T-image</u>

i.e. $TA : Prop$ for each type A

$TB \to B$ for $B : Prop$

<u>Idea</u>:

The logical strength of the rules for the type Prop is given by:

> There is a strict lax modality T such that Prop is a T-image.

Moreover adding $(\Pi p : Prop)[\neg\neg p \to p]$ does not increase its logical strength and then $\neg\neg$ is the unique such T, up to logical equivalence.

Back to full impredicativity

Theorem (Aczel 1999)**.** There is a proof-theoretical equivalence

$$\mathsf{MLWPropU}_{\leq\omega} \equiv \mathsf{IZF}^{\neg\neg}\mathsf{u}_{\leq\omega}$$

Idea: replace the Russell-Prawitz modality with double negation.

Theorem (Rathjen 2012)**.** There is a proof-theoretical equivalence

$$\mathsf{MLVProp} \equiv \mathsf{KP}(\mathcal{P})$$

New questions:

- Find setting to explore representations of logic in type theory
- Relate reinterpretations of logic in set theory and in type theory

Motivation also from Categorical Logic, cf. Maietti's PhD thesis 1998.

Sets and categories

"There seems to be a great deal of difficulty in having a good discussion on foundations between set theorists (and other logicians who are persuaded to some extent by the cumulative hierarchy picture) and category theorists (who believe that the notion of topos is important for foundations).

I do not feel myself to be entirely located on one side or the other of the discussion. But I would like to understand the issues better. So here are my thoughts, written rather hastily. I try to describe the discussion, as it appears to me.

I may have got it seriously wrong. I am aware that there are aspects that I have not covered properly."

P. Aczel, email to the F.O.M. Mailing List, January 1998.

Predicative Algebraic Set Theory

Papers by Moerdijk and Palmgren linked CZF with categories ($\sim$1998):

- I. Moerdijk and E. Palmgren, Wellfounded trees in categories
- I. Moerdijk and E. Palmgren, Type theories, toposes and Constructive Set Theory: predicative aspects of AST

See also work of B. van den Berg and I. Moerdijk.

Key aspects:

- inspired by the type-theoretic interpretation,
- use propositions-as-subobjects, not propositions-as-types
- possibility of developing internal sheaf models
- new axioms for constructive set theories (AMC)

Part II:
Logic-enriched type theories

Logic-enriched type theories

Talk at TYPES 1999.

These extend Martin-Löf type theories with additional judgements

$$(\Gamma)\ \phi : \mathsf{prop} \qquad (\Gamma)\ \phi_1, \ldots, \phi_n \vdash \phi$$

One can then formulate deduction rules for:

- dependently-typed intuitionistic logic
- induction principles
- classical logic

without modifying the underlying type theory.

The propositions-as-types interpretation

$\mathbf{ML} + \mathbf{IL}$ admits a prop-as-types interpretation into $\mathbf{ML}$:

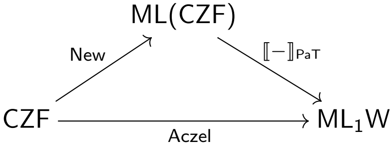
$$\phi : \mathsf{prop} \qquad \rightsquigarrow \qquad [\![\phi]\!]_{\mathsf{PaT}} : \mathsf{type}$$

Proposition. The following are equivalent:

1. $\mathbf{ML} + \mathbf{IL} + (\mathbf{AC}) + (\mathbf{0}\bot) \vdash \phi$
2. $\mathbf{ML} \vdash p : [\![\phi]\!]_{\mathrm{PaT}}$, for some p.

A factorisation of the type-theoretic interpretation

Talk at TYPES 2000.



Key ideas: introduce 'collection principles' in type theory that

- are valid under $[\![-]\!]_{\mathsf{PaT}}$
- are preserved by j-reinterpretations of logic (cf. sheaf models)

Reinterpretations of logic

Definition. A *topology* is an operator $j : \mathsf{P} \to \mathsf{P}$ such that:

1. $p \leq j(p)$
2. $p \leq q \vdash j(p) \leq j(q)$,
3. $j(p) \wedge j(q) \leq j(p \wedge q)$,
4. $j(j(p)) \leq j(p)$

Define

$$J(\varphi) = (\exists p : \mathsf{P})\big(\, j(p) \wedge (p \to \varphi)\,\big)$$

This determines a reinterpretation of logic.

Theorem.

1. The j-interpretation of each rule of $\mathrm{ML}(\mathrm{CZF}^{-})$ is sound.
2. If j is set presented, then the j-interpretation of the Subset Collection rule is sound.

Part III:
HoTT

Timeline

- Feb 2010 - Sep 2011: Voevodsky's Univalent Foundations library
- December 2010: Peter and I became aware of the library
- January 2011: discussions on translating the library to paper-and-pencil mathematics
- February 27th - March 5th, 2011: Oberwolfach Meeting
- September 2012 - June 2013: Special Year in Princeton

Some of Peter's contributions to HoTT

- The identity type weak factorisation system: connection to Hoffmann's work on Paulin-Mohring's rule
- Structure Identity Principle (Coquand)
- HoTT book (e.g. identity systems)

Homotopy Type Theory

Some additional principles:

(FE) Function extensionality
(TR) $||-||_n$-truncations
(UA) Univalence Axiom
(Q) Set quotients

Some homotopy type theories:

- **H** $=$ **ML** $+$ **(FE)** $+$ **(TR)**
- **HoTT** $=$ **(H)** $+$ **(UA)** $+$ **(Q)**

The propositions-as-hprop interpretation

Logic-enriched type theories admit also a ‘prop-as-hprop’ intepretation into homotopy type theories.

$$\phi : \mathsf{prop} \qquad \rightsquigarrow \qquad [\![\phi]\!]_{\mathsf{PhP}} : \mathsf{hprop}$$

Consider a logic-enriched version $\mathsf{H} + \mathsf{L}$ of H.

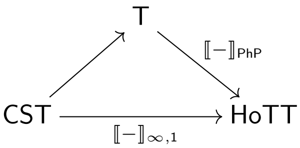
Proposition (Gallozzi 2021)**.** The following are equivalent.

- $\mathsf{H} + \mathsf{L} + \mathbf{AC}_! \vdash J$
- $\mathsf{H} \vdash [\![J]\!]_{\mathsf{PhP}}$

where $\mathbf{AC}_!$ is the axiom of unique choice.

Factorisation via logic-enriched type theories

Theorem (Gallozzi 2021)**.** There is a factorisation



where T is a logic-enriched type theory.

Key ideas: introduce axioms that

- are valid under $[\![-]\!]_{\mathsf{PhP}}$
- suffice to validate CST.

Some references

- P. Aczel
  On relating type theories and set theories
  TYPES 1998.

- P. Aczel
  The Russell-Prawitz modality
  Mathematical Structures in Computer Science, 2001.

- P. Aczel and N. Gambino
  Collection Principles in Dependent Type Theory
  TYPES 2000.

Peter Aczel Memorial Meeting

University of Manchester, 10th September 2025

Speakers:

1. Steve Awodey (Carnegie Mellon University)
2. Rosalie Iemhoff (Utrecht University)
3. Andrew Swan (University of Ljubljana)
4. Jouko Väänänen (University of Helsinki)